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Dispersiveness and Positive Contractive Semigroups

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The generators of positive C_0- and C_0^*-semigroups on an ordered Banach space are characterized in case the positive cone is normal. The notion of dispersiveness is introduced and conditions are given in order that dispersiveness of the generator corresponds to positivity and contractivity of the semigroup. As an application an order-theoretic description of the generating derivations on a C^*-algebra is given. Finally two distinct characterizations of the generators of strongly continuous unitary groups on a real Hilbert space keeping invariant a closed convex cone are given.

INTRODUCTION

In recent years the study of ordered Banach spaces and positive one-parameter semigroups has been intensified. The problem of characterizing the generators of positive semigroups was considered in the C^*-algebra setting by Evans and Hanche-Olsen [8] and by Bratteli and Robinson [6]. In [1] the problem was abstracted and the generators of positive C_0-semigroups was described when the positive cone has nonempty interior. This case was also investigated in [7]. In the papers [11–13] Robinson and Yamamuro studied the order structure of ordered Banach spaces and positive semigroups were considered in an abstract setting in [2, 14, 15]. The results obtained so far have finally been summarized in [3].

In the two first sections of this paper we extend some results on the generation of positive C_0- and C_0^*-semigroups to the case where the positive cone is only assumed to be normal. Our proofs do not rely on earlier results. We also introduce the notion of dispersiveness in the more abstract setting and give a condition on the positive cone in order that dispersiveness corresponds to positivity and contractivity of the generated semigroup.

In Section 3 we apply the results to C^*-algebras and obtain a characterization of the generators of C_0-groups of automorphisms without reference to derivations.

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In the final Section 4 we consider a real Hilbert space ordered by an arbitrary closed cone. In this setting we characterize the generators of both positive contractive C_0-semigroups and of positive C_0-groups of unitaries.

1. Definitions, Notation and Preliminary Lemmas

An *ordered Banach space* $(X, X_+, \|\cdot\|)$ consists of a real Banach space X with norm $\|\cdot\|$ and a norm-closed convex cone $X_+ \subseteq X$.

If X^* denotes the dual Banach space with norm $\|\cdot\|^*$ we can define a weak* closed convex subset $X_+^* \subseteq X^*$ by

$$X_+^* = \{l \in X^* \mid l(x_+) \geqslant 0 \text{ for all } x_+ \in X_+\}.$$

An *ordered Banach dual space* $(X, X_+, \|\cdot\|)$ is an ordered Banach space which arises in this way from another ordered Banach space $(X_*, X_*^+, \|\cdot\|_*)$.

Given an ordered Banach space $(X, X_+, \|\cdot\|)$ we define an order relation by setting $x \geqslant y$ whenever $x - y \in X_+$.

X_+ is *normal* if for some constant $\alpha \geqslant 1$, $x \leqslant y \leqslant z$ implies that $\|y\| \leqslant \alpha(\|x\| \vee \|z\|)$. To emphasize the constant we call X_+ α-normal in this case.

X_+ is *generating* if for some $\alpha \geqslant 1$ every $x \in X$ admits a decomposition $x = x_+ - x_-$ with $x_+, x_- \in X_+$ and $\|x_+\| + \|x_-\| \leqslant \alpha \|x\|$. We call X_+ α-generating in this case.

If the elements of the form $x_+ - x_-$, $x_+, x_- \in X_+$ are norm-dense in X we say that X_+ is *weakly generating.*

Throughout the first two sections we assume X_+ is *proper*, i.e., that $X_+ \cap (-X_+) = \{0\}$.

$\|\cdot\|$ is *monotone* when $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant \|y\|$.

The space $\mathscr{L}(X)$ consisting of the bounded operators on X becomes an ordered Banach space when we let

$$\mathscr{L}_+(X) \equiv \{S \in \mathscr{L}(X) \mid S(X_+) \subseteq X_+\}$$

be the positive cone in $\mathscr{L}(X)$ and endow it with the usual operator norm. We say that the operator norm on $\mathscr{L}(X)$ is *positively attained* if $\|S\| = \sup\{\|Sx\| \mid x \geqslant 0, \|x\| \leqslant 1\}$ for all $S \in \mathscr{L}_+(X)$. If $(X, X_+, \|\cdot\|)$ is an ordered Banach dual space we say that the operator norm on $\mathscr{L}(X)$ is *weakly positively attained* if it is positively attained for weak*-continuous elements of $\mathscr{L}_+(X)$.

In the present context the notion of positive attainment seems to originate in [14]. It is explicitly defined in [3] and criteria for positive attainment are given in [3]. Another reference is [16].

Note that all the defined properties are actually intrinsic to the triple $(X, X_+, \|\cdot\|)$ rather than to any of the three alone.

It is a well-known and useful fact that X_+ is α-normal if and only if the dual cone X_+^* is α-generating [3, Theorem 1.1.4].

To simplify notation we put

$$M = X_1^* \cap X_+^* = \{l \in X^* \mid l \in X_+^* \text{ and } \|l\|^* \leqslant 1\}$$

and in case of an ordered Banach dual space

$$M_* = X_*^1 \cap X_*^+ = \{l \in X_* \mid l \in X_*^+, \|l\|_* \leqslant 1\}.$$

Define $N(x) = \max_{l \in M} l(x)$ and

$$\|x\|_N = N(x) \vee N(-x) = \max_{l \in M} |l(x)| \qquad \text{for} \quad x \in X.$$

Note that $\|x\|_N \leqslant \|x\|$ for all $x \in X$ and that $\|\cdot\|_N$ is a norm on X exactly when X_+ is proper.

By [3, Proposition 1.6.2] $N(\cdot)$ is the canonical half-norm corresponding to X_+:

$$N(x) = \inf\{\|x + y\| \mid y \in X_+\} = \inf\{\|y\| \mid y \in X, y \geqslant x\}.$$

$N(\cdot)$ measures the distance from $-x$ to X_+.

If $(X, X_+, \|\cdot\|)$ is an ordered Banach dual space

$$N(x) = \sup_{l \in M_*} l(x)$$

again by [3, Proposition 1.6.2]. As a consequence M_* is weak* dense in M by the Hahn–Banach separation theorem.

Though simple the following four lemmas are actually the cornerstone for the results of this paper.

LEMMA 1.1. *X_+ is α-normal if and only if $\|x\| \leqslant \alpha \|x\|_N$ for all $x \in X$.*

Proof. If X_+ is a α-normal X_+^* is α-generating. Thus if $l \in X$ and $\|l\|^* \leqslant 1$ we have $l = l_+ - l_-$ and $\|l_+\|^* + \|l_-\|^* \leqslant \alpha$. Therefore

$$|l(x)| \leqslant |l_+(x)| + |l_-(x)| \leqslant \|l_+\|^* \|x\|_N + \|l_-\|^* \|x\|_N \leqslant \alpha \|x\|_N$$

for all $x \in X$.

Assume next $\|x\| \leqslant \alpha \|x\|_N$ for all $x \in X$ and let $x \leqslant y \leqslant z$ with $\|x\| \vee \|z\| \leqslant 1$.

If $l \in M$

$$-1 \leqslant l(x) \leqslant l(y) \leqslant l(z) \leqslant 1 \qquad \text{so} \quad \|y\|_N \leqslant 1.$$

Thus

$$\|y\| \leqslant \alpha \|y\|_N \leqslant \alpha.$$

In [1] it is mentioned that X_+ is normal if and only if $\|\cdot\|$ and $\|\cdot\|_N$ are equivalent. The new feature in Lemma 1.1 is the role of α.

A version of the next lemma appears in [11].

LEMMA 1.2. *Let $l \in X^*$. Then the following conditions are equivalent:*

(a) $l(x) \leqslant N(x)$ *for all* $x \in X$.

(b) $l \in X_+^*$ *and* $|l(x)| \leqslant \|x\|_N$ *for all* $x \in X$.

(c) $l \in M$.

Proof. (a) $\Rightarrow$ (b) If $x \in X_+$ we have $l(-x) \leqslant N(-x) = 0$ so $l(x) \geqslant 0$. Thus $l \in X_+^*$. For general x we still have $l(-x) \leqslant N(-x)$ or $l(x) \geqslant -N(-x)$. Therefore

$$-\|x\|_N = -(N(x) \vee N(-x)) \leqslant l(x) \leqslant N(x) \vee N(-x) = \|x\|_N$$

(b) $\Rightarrow$ (c) is immediate since $\|x\|_N \leqslant \|x\|$

(c) $\Rightarrow$ (a) is immediate from the definition of $N(\cdot)$.

Also the next lemma can be found in another version, see [13].

LEMMA 1.3. *Let A be a bounded operator on X. Then the following conditions are equivalent:*

(a) *A is $N(\cdot)$-contractive.*

(b) *A is positive and $\|\cdot\|_N$-contractive.*

(c) $A^*(M) \subseteq M$.

Proof. (a) $\Rightarrow$ (b) If $x \in A_+$ we have $0 \leqslant N(A(-x)) \leqslant N(-x) = 0$. So $Ax \in A_+$ and A is positive. For general x in X

$$\|Ax\|_N = N(Ax) \vee N(-Ax) \leqslant N(x) \vee N(-x) = \|x\|_N$$

so (b) is established.

(b) $\Rightarrow$ (c) Let $l \in M$ and $x \in X$. Then

$$|(A^*l)(x)| = |l(Ax)| \leqslant \|Ax\|_N \leqslant \|x\|_N \leqslant \|x\|.$$

So $\|A^*l\| \leqslant 1$ and since A is positive $A^*l \in M$.

(c) $\Rightarrow$ (a) is immediate from the definition of $N(\cdot)$.

LEMMA 1.4. *Assume $(X, X_+, \|\cdot\|)$ is an ordered Banach dual space and A a bounded weak* continuous operator. Then the following conditions are equivalent*:

(a) *A is $N(\cdot)$-contractive.*

(b) *A is positive and $\|\cdot\|_N$-contractive.*

(c) $A_*(M_*) \subseteq M_*$.

Proof. The proof is quite analogous to the preceding.

2. Positive One-Parameter Semigroups

A one-parameter semigroup $\{S_t\}_{t\geqslant 0}$ on an ordered Banach space $(X, X_+, \|\cdot\|)$ is a positive semigroup if $S_t \in \mathscr{L}_+(X)$ for all $t \geqslant 0$.

We will consider both C_0-semigroups and C_0^*-semigroups and refer the reader to [5] for definitions and theory for such semigroups.

THEOREM 2.1. *Let $(X, X_+, \|\cdot\|)$ be an ordered Banach (dual) space. Assume X_+ is α-normal. Let H be an operator on X. Then the following conditions are equivalent*

(1) *H generates a positive $C_0(C_0^*)$-semigroup.*

(2) *H is norm- (weak*-) densely defined, norm- (weak*-) closed with*

$$R(I - t_0 H) = X \qquad \textit{for some} \quad t_0 > 0,$$

and

$$N([I - tH]^n x) \geqslant (1 - tj)^n M^{-1} N(x)$$

for all $0 < t \leqslant t_0$, $x \in D(H^n)$, and $n \geqslant 1$, for some $M \geqslant 1, j \geqslant 0$ with $t_0 j < 1$.

In this case $\|e^{tH}\|_N \leqslant Me^{jt}$ and $\|e^{tH}\| \leqslant \alpha Me^{jt}$ for all $t \geqslant 0$.

Proof. (1) $\Rightarrow$ (2) Since $\|\cdot\|_N$ is an equivalent norm on X it follows from semigroup theory that $(I - tH)^{-n}$ exists for all $t \geqslant 0$, $n \geqslant 1$ as bounded (and weak* continuous) positive operators with

$$\|(I - tH)^{-n}\|_N \leqslant (1 - tj)^{-n} M$$

for some $M \geqslant 1$, $j \geqslant 0$ when $tj < 1$. So (2) follows from Lemma 1.3 (Lemma 1.4).

(2) $\Rightarrow$ (1) By definition of $\|\cdot\|_N$ (2) yields

$$\|(I - tH)^n x\|_N \geqslant (1 - tj)^n M^{-1} \|x\|_N$$

for the same t, j, n, M and as in (2).

Thus the Feller–Miyadera–Philips theorem, see e.g., [3], yields that H generates a $C_0(C_0^*)$-semigroup on X. If $x \geqslant 0$ (2) gives

$$N(-(I-tH)^{-1}x) \leqslant M(1-tj)^{-1} N(-x) = 0$$

for t small enough. So $(I-tH)^{-1}x \in X_+$ and $(I-tH)^{-1} \in \mathscr{L}_+(X)$ for all small t. Thus the generated semigroup is positive since $e^{tH}x = \lim_{n\to\infty}(I-tH/n)^{-n}x$ for $t \geqslant 0$ and $x \in X$, where the limit exists in the right topology by semigroup theory.

$\|e^{tH}\|_N \leqslant Me^{jt}$ follows from the Feller–Miyadera–Philips theorem. The last norm estimate follows from Lemma 1.1.

Theorem 2.1 is another version of results given in [14] and [15]. It improves these results by replacing monotonicity of $\|\cdot\|$ by normality of X_+. However, it contains an asymmetry. When H generates a semigroups with the bound $\|e^{tH}\| \leqslant Ne^{jt}$ it satisfies (2) with $M = \alpha N$. But if it satisfies (2) with $M = \alpha N$ we can only deduce that the generated semigroup satisfies $\|e^{tH}\| \leqslant \alpha^2 Ne^{jt}$. This asymmetry would be removed if

$$\|S\|_N = \|S\| \qquad \text{for} \quad S \in \mathscr{L}_+(X). \tag{A}$$

Especially for the purpose of using Theorem 2.1 to characterize the generators of positive *and* contractive semigroups it becomes useful to have (A). Property (A) does not appear to have been investigated in the literature.

Since Lemma 1.3 implies that we always have $\|S\|_N \leqslant \|S\|$ for $S \in \mathscr{L}_+(X)$, it is enough to worry about the opposite inequality.

LEMMA 2.2. *$\mathscr{L}(X)$ satisfies* (A) *if and only if $\mathscr{L}(X^*)$ is weakly positively attained.*

Proof. Assume $\|S\| \leqslant \|S\|_N$ for all $S \in \mathscr{L}_+(X)$ and let $T \in \mathscr{L}_+(X^*)$ be a weak*-continuous operator.

Put $\|T\|_+ = \sup\{\|Tl\|_* \mid l \in M\}$. Clearly $\|T\|_+ \leqslant \|T\|$ and we must show the opposite inequality. $T = S^*$ for some $S \in \mathscr{L}_+(X)$ and since $(S^*/\|T\|_+)(M) \subseteq M$ Lemma 1.3 implies that $\|S\|_N \leqslant \|T\|_+$. By assumption $\|S\| \leqslant \|S\|_N$ and since $\|T\| = \|S\|$ we have $\|T\| = \|T\|_+$.

By Lemma 1.3 $(S^*/\|S\|_N)(M) \subseteq M$ when $S \in \mathscr{L}_+(X)$ and if $\mathscr{L}(X^*)$ is weakly positively attained we conclude that $\|S\| = \|S^*\| \leqslant \|S\|_N$. Q.E.D.

It is easy to see that if the implication

$$\left.\begin{array}{l} N(x) \leqslant N(y) \\ N(-x) \leqslant N(-y) \end{array}\right\} \Rightarrow \|x\| \leqslant \|y\|, \qquad x, y \in X \tag{B}$$

holds, then $\mathscr{L}(X)$ has property (A). Property (B) occurs in [3, Sect. 2.1].

In the opposite direction if $\mathscr{L}(X)$ has property (A) and $x, y \in X_+$ then $N(x) \leqslant N(y)$ implies $\|x\| \leqslant \|y\|$. To see this use the Hahn–Banach theorem to choose a linear functional l such that $l(z) \leqslant N(z)$ for all $z \in X$ and $l(y) = N(y)$. The proof of Lemma 1.2 establishes that $l \in M$. Define the operator S by $Sz = (l(z)/N(y))x$ (if $N(y) = 0$ both x and y are zero). Clearly $S \in \mathscr{L}_+(X)$ and $\|S\|_N \leqslant 1$. Therefore $\|S\| \leqslant 1$ and

$$\|x\| = \|Sy\| \leqslant \|y\|$$

since $\mathscr{L}(X)$ has property (A).

In particular this shows that if $\mathscr{L}(X)$ has property (A) then $\|\cdot\|$ is monotone and X_+ 3-normal.

Lemma 2.3. *Each of the following conditions imply that $\mathscr{L}(X)$ has property* (A):

(1) X_+ *has property* (B).

(2) X_+ *is* 1-*generating and* $\|\cdot\|$ *is monotone.*

(3) *The operator norm on $\mathscr{L}(X)$ is positively attained and* $\|\cdot\|$ *is monotone.*

Proof. We have already mentioned (1). Assume (2) holds. Then $N(x) = \|x\|$ for $x \in X_+$ by [3, Theorem 1.6.3] and every $z \in X$ decomposes as $z = z_+ - z_-$ where $z_+, z_- \in X_+$ and $\|z\| = \|z_+\| + \|z_-\|$. Thus if $N(Sx) \leqslant N(x)$ for all $x \in X$ we have

$$\begin{aligned}\|Sz\| &\leqslant \|Sz_+\| + \|Sz_-\| = N(Sz_+) + N(Sz_-) \leqslant N(z_+) + N(z_-)\\ &= \|z_+\| + \|z_-\| = \|z\|.\end{aligned}$$

Thus $\|S\| \leqslant 1$.

Assume (3) holds and $N(Sx) \leqslant N(x)$ for all $x \in X$. If $x_+ \in X_+$ we have

$$\|Sx_+\| = N(Sx_+) \leqslant N(x_+) = \|x_+\|.$$

Since $\mathscr{L}(X)$ is assumed to have positively attained norm $\|S\| \leqslant 1$.

Definition. Let H be an operator on an ordered Banach space $(X, X_+, \|\cdot\|)$. H is *dispersive* if $N((I - tH)x) \geqslant N(x)$ for all $t \geqslant 0$ and $x \in D(H)$.

Alternatively [1] H is dispersive if every $x \in D(H)$ admits an $l \in M$ such that $l(x) = N(x)$ and $l(Hx) \leqslant 0$. In [1] it was proved that if H is dispersive and norm-densely defined then $l(Hx) \leqslant 0$ whenever $x \in D(H)$ and $l \in M$ satisfies $l(x) = N(x)$.

THEOREM 2.4. *Let $(X, X_+, \|\cdot\|)$ be an ordered Banach space. Assume $\mathscr{L}(X)$ satisfies property* (A). *Let H be an operator on X. Then H generates a positive C_0-semigroup of contractions if and only if*

(1) *H is norm-densely defined and closed.*

(2) *H is dispersive.*

(3) *$R(I-tH)=X$ for some $t>0$ or (3′) H^* is dispersive (if X_+ is weakly generating).*

Proof. The necessity and sufficiency of (1), (2), and (3) follows from Theorem 2.1 and the discussion above.

The necessity of (3′) follows by considering the dual semigroup $\{e^{tH^*}\}_{t\geq 0}$: For $l \in X^*$ we have

$$N(e^{tH^*}l) = \sup\{e^{tH^*}l(x) \mid x \in X_+ \cap X_1\}.$$

But $\|\cdot\|$ is monotone by the above discussion so $\|x\| = N(x)$ for $x \in X_+$ [3, Theorem 1.6.3].

Thus

$$\begin{aligned} N(e^{tH^*}l) &= \sup\{e^{tH^*}l(x) \mid x \geq 0, N(x) \leq 1\} \\ &= \sup\{l(e^{tH}x) \mid x \geq 0, N(x) \leq 1\} \\ &\leq \sup\{l(x) \mid x \geq 0, N(x) \leq 1\} \end{aligned}$$

since $e^{tH} \in \mathscr{L}_+(X)$ and $\|e^{tH}\|_X \leq 1$.

Thus e^{tH^*} is N-contractive. By the arguments used to prove (1) ⇒ (2) in Theorem 2.1 ($M=1$, $j=0$ in this case) we can conclude that H^* is dispersive. If we assume (1), (2), and (3′) it follows from (3′) that $\ker(I-tH^*)=\{0\}$ or that $R(I-tH)$ is norm-dense in X. But dispersiveness and closedness of H imply that we must have $R(I-tH)=X$. The proof is complete.

Theorem 2.4 is a generalization to dispersive operators and positive semigroups of an old result of Lumer and Phillips. These authors were the first to remark that the range condition (3) can be replaced by dissipativity of the adjoint.

COROLLARY 2.5. *Let $(X, X_+, \|\cdot\|)$ be an ordered Banach dual space. Assume the operator norm on $\mathscr{L}(X)$ is weakly positively attained. Then an operator H on X generates a positive C_0^*-semigroup of contractions if and only if*

(1) *H is a weak*-continuous and weak*-closed operator.*

(2) *H is dispersive.*

(3) *H_* is dispersive.*

If we instead assume $\mathscr{L}(X)$ has property (A) *we have that H generates a positive C_0^*-semigroup of contractions if and only if H satisfies* (1), (2), *and that $R(I-tH)=X$ for some $t>0$.*

Proof. The first statement follows from Theorem 2.4 and Lemma 2.2. The second follows from Theorem 2.1.

3. An Application to C^*-Algebras

Definition 3.1. Let $(Z, \|\cdot\|)$ be a complex Banach space and $i: Z \to Z$ a complex continuous involution, i.e.,

$$\begin{aligned} i(\lambda z_1 + z_2) &= \bar{\lambda} i(z_1) + i(z_2), \qquad \lambda \in \mathbb{C}, \quad z_1, z_2 \in Z, \\ i(i(z)) &= z, \\ \|i(z)\| &= \|z\|, \qquad z \in Z. \end{aligned}$$

Let $Z_{\mathrm{R}} = \{z \in Z \mid i(z) = z\}$. Z_{R} is a real Banach space and if $Z_+ \subseteq Z_{\mathrm{R}}$ is a norm-closed convex cone we call $(Z, Z_+, \|\cdot\|, i)$ a *complex ordered Banach space.*

All properties of the preceding sections can be defined in a natural way in the complex setting. For our purposes we only need the following:

Definition 3.2. An operator H on a complex ordered Banach space is called *dispersive* if $i(D(H)) = D(H)$, $i(Hz) = Hi(z)$ for $z \in D(H)$, and $H|_{Z_{\mathrm{R}}}$ is dispersive.

With these definitions we can formulate the next theorem.

Theorem 3.3. *Let $\mathscr{A}$ be a C^*-algebra and δ an operator on $\mathscr{A}$. Then δ generates a C_0-group of $*$-automorphisms on $\mathscr{A}$ if and only if*

(1) *δ is norm-densely defined and norm-closed.*

(2) *$\pm\delta$ and $\pm\delta^*$ are all dispersive.*

Proof. By the above definitions and Theorem 2.4 (1) and (2) are certainly necessary conditions. If (1) and (2) hold Theorem 2.4 ensures that δ generates a C_0-group $\{\varphi_t\}_{t\in\mathbb{R}}$ of positive isometries. By [5, Corollary 3.2.13] $\{\varphi_t\}_{t\in\mathbb{R}}$ consists of $*$-automorphisms.

The corresponding theorem holds for C_0^*-groups of Jordan automorphisms on a W^*-algebra $\mathscr{M}$. Only if $\mathscr{M}$ is a factor or abelian can one conclude that the generated group consists of $*$-automorphisms. In general it is impossible as shown by a counterexample in [4].

4. Positive Semigroups in a Hilbert Space Ordered by a Closed Cone

Let $\mathscr{H}$ denote a real Hilbert space and $\mathscr{P} \subseteq \mathscr{H}$ a closed convex cone. Let $\mathscr{P}^0$ denote the dual cone, i.e.,

$$\mathscr{P}^0 = \{\psi \in \mathscr{H} \mid \langle \psi, \varphi \rangle \geqslant 0 \ \forall \varphi \in \mathscr{P}\}.$$

Then every element φ in $\mathscr{H}$ admits a unique decomposition $\varphi = \varphi_+ - \varphi_-$ where $\varphi_+ \in \mathscr{P}$, $\varphi_- \in \mathscr{P}^0$, and $\langle \varphi_+, \varphi_- \rangle = 0$. Furthermore $\|\psi_+ - \varphi_+\| \leqslant \|\psi - \varphi\|$ for all ψ, φ in $\mathscr{H}$.

The proof of this fact follows known lines and a omitted. See [9, I.1.2. Lemma].

The signs $+, -$ as subscripts will refer to such a decomposition in this section.

If $\mathscr{P} = \mathscr{P}^0$, i.e., $\mathscr{P}$ is a selfdual cone we can apply the results of Section 2 directly. However, the ideas can also be used to obtain results in the more general setting:

Proposition 4.1. *Let $\{e^{tS}\}_{t \geqslant 0}$ be a strongly continuous contractive semigroup on $\mathscr{H}$. Then the following conditions are equivalent:*

(a) $e^{tS}\mathscr{P} \subseteq \mathscr{P}$ *for all* $t \geqslant 0$.

(b) $\langle S\varphi, \varphi_- \rangle \geqslant 0$ *for all* φ *in* $D(S)$.

Proof. (a) $\Rightarrow$ (b)

$$\langle S\varphi, \varphi_- \rangle = \lim_{t \to 0_+} \frac{1}{t} (\langle e^{tS}\varphi, \varphi_- \rangle - \langle \varphi, \varphi_- \rangle)$$

$$= \lim_{t \to 0_+} \frac{1}{t} (\langle e^{tS}\varphi_+, \varphi_- \rangle - \langle e^{tS}\varphi_-, \varphi_- \rangle + \|\varphi_-\|^2).$$

So $\langle S\varphi, \varphi_- \rangle \geqslant 0$ since $\langle e^{tS}\varphi_+, \varphi_- \rangle \geqslant 0$ and $\langle e^{tS}\varphi_-, \varphi_- \rangle \leqslant \|\varphi_-\|^2$ for all $t \geqslant 0$.

(b) $\Rightarrow$ (a) Let $\psi \in \mathscr{P}$ and $\alpha \geqslant 0$. Put $\varphi = (I - \alpha S)^{-1}\psi$. Then $\|\varphi_-\|^2 = -\langle \varphi, \varphi_- \rangle \leqslant -\langle \varphi, \varphi_- \rangle + \alpha \langle S\varphi, \varphi_- \rangle = -\langle \psi, \varphi_- \rangle \leqslant 0$. So $\varphi_- = 0$ and we see that $(I - \alpha S)^{-1}\mathscr{P} \subseteq \mathscr{P}$.

Since $e^{tS}\psi = \lim_{n \to \infty} (I - (tS^{-n}/n))\psi$ the proof is complete.

Proposition 4.1 is a version of a result of Beurling and Deny. See [10, pp. 208–212].

The main result of this section is

Theorem 4.2. *Let $\{e^{tT}\}_{t \in \mathbb{R}}$ be a strongly continuous unitary group on $\mathscr{H}$. Then the following conditions are equivalent:*

(1) $e^{tT}\mathscr{P} \subseteq \mathscr{P}$ *for all* $t \in \mathbb{R}$.

(2) (a) $\varphi \in D(T) \Rightarrow \varphi_+, \varphi_- \in D(T)$.

(b) *If* $\varphi \in D(T) \cap \mathscr{P}$, $\psi \in \mathscr{P}^0$, *and* $\langle \varphi, \psi \rangle = 0$ *then* $\langle T\varphi, \psi \rangle = 0$.

(3) $\langle T\varphi, \varphi_- \rangle = 0$ *for all* $\varphi \in D(T)$.

Proof. The equivalence of (1) and (3) is an easy consequence of Proposition 4.1.

(1) $\Rightarrow$ (2) For every φ in $\mathscr{H}$

$$\|e^{tT}\varphi_+ - \varphi_+\| \leqslant \|e^{tT}\varphi - \varphi\|$$

since $\{e^{tT}\}_{t \in \mathbb{R}}$ is $\mathscr{P}$-positive and unitary. So (a) is a consequence of the fact that an element ψ in $\mathscr{H}$ is in $D(T)$ if and only if

$$\sup_{t \neq 0} \frac{\|e^{tT}\psi - \psi\|}{|t|} < \infty$$

[5, Proposition 3.1.23]. This proves (a).

Let ψ and φ be as in (b). Then

$$\langle T\varphi, \psi \rangle = \frac{d}{dt} \langle e^{tT}\varphi, \psi \rangle|_{t=0}$$

and the function $t \to \langle e^{tT}\varphi, \psi \rangle$ has a global minimum at 0. So $\langle T\varphi, \psi \rangle = 0$.

(2) $\Rightarrow$ (1) Let $\psi \in \mathscr{P}$. As in the proof of Proposition 4.1 it is enough to prove that $(t \pm T)^{-1}\psi \in \mathscr{P}$ for all positive t.

Choose $t > 0$ and put $\varphi = (t+T)^{-1}\psi$. $(t+T)\varphi = \psi \in \mathscr{P}$ and $\varphi_- \in \mathscr{P}^0$ so

$$\begin{aligned} 0 \leqslant \langle (t+T)\varphi, \varphi_- \rangle &= \langle (t+T)\varphi_+, \varphi_- \rangle - \langle (t+T)\varphi_-, \varphi_- \rangle \\ &= -t\|\varphi_-\|^2 - \langle T\varphi_-, \varphi_- \rangle \end{aligned}$$

But $T^* = -T$ so $\langle T\varphi_-, \varphi_- \rangle = 0$ and hence $0 \leqslant -t\|\varphi_-\|^2$ which is only possible if $\varphi_- = 0$.

The proof that $(t-T)^{-1}\psi \in \mathscr{P}$ is the same.

As shown by [3, Example 1.1.10] the cones considered in this section need not be normal.

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